Subject: Physics

Production of Courseware
e-Content for Post Graduate Courses

Paper No. : Quantum Mechanics-I
Module : Heisenberg's Uncertainty Principle

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Description of Module	
Subject Name	Physics
Paper Name	Quantum Mechanics-I
Module Name/Title	Heisenberg's Uncertainty Principle
Module Id	04

Contents of this Unit

Learning Outcomes

After studying this module, you shall be able to

- Know how the superposition of plane harmonic waves of different wavelengths or wave numbers can produce a wave packet localized in space
- Learn an important criterion for localizing a wave packet describing a particle in space
- Know the superposition of plane harmonic waves of varying frequencies or angular frequencies can produce a localized wave packet in time
- Define the group velocity of a wave packet and show that it is in agreement with the classical expression of a particle
- Analyze the evolution of a wave packet in time and show that the spread of the wave packet increases with time

1.INTRODUCTION

This module, which is in continuation of the preceding one, introduces the concepts such as Heisenberg's uncertainty principle and the complementary principle enunciated by Bohr. In the preceding module while discussing the wave packets in space and time, we found that there are fundamental limitations placed by the wave packet picture on the accuracy with which both the position and momentum of a particle can be specified. In fact from the theory of Fourier transform it can be shown quite generally that if a wave packet in space has a spread Δx, then the Fourier transform must be non vanishing over a range of width Δk (k being the wave number of harmonic waves) with $\Delta k \geq (1/\Delta x)$. In other words, the spread in momenta (Δp) of the de-Broglie waves contributing to wave packet satisfies an inequality (Δp) (Δx) $\geq \hbar$. Similar inequality between energy and time also holds when we consider a wave packet in time constructed from the superposition of a spectrum of harmonic waves in frequencies (or angular frequencies). This is the essential content of the uncertainty principle, which we shall be studying in the present module. The underlying idea that the basic properties of matter, such as momentum and position, do not exist, in quantum theory, in a precisely defined form constitute a far-reaching change that led Bohr to enunciate it in terms of what he called the complementary principle. This will be illustrated through the example of a double slit experiment in the present module.

2.HEISENBRG'S UNCERTAINTY PRINCIPLE:

One of the important contents of the theory of quantum mechanics is the uncertainty principle, developed by W. Heisenberg in 1927. Stated in simple terms, it says that if the x-coordinate of the position of a quantum mechanical particle is known to an accuracy, Δx, then the x-component of the corresponding momentum variable cannot be determined to an accuracy better than $\Delta p_x \approx h/\Delta x$. In other words, one cannot determine precisely and simultaneously the values of both the members of particular pairs of physical variables, known as canonically conjugate to each other, which define a given atomic system. The x- (or y- or z-) component of the position coordinate of a particle and the corresponding component of momentum, p_x (or p_y $or\ p_z$) , a component L_z of angular momentum and its angular coordinate ϕ (in the xy-plane), the energy E of a particle and the time t at which the energy is measured – are some of the examples of canonically conjugate pairs. Thus, expressed in quantitative terms, the uncertainty principle states that,

$$\Delta x\,\Delta p_X \geq \hbar, \qquad \Delta L_Z\,\Delta\phi \geq \hbar, \qquad \Delta E\,\Delta t \geq \hbar \tag{4.1}$$

The fact that h is so small makes the uncertainty principle relevant primarily to the systems of atomic size.

The formal derivation of Heisenberg's Uncertainty principle along with the precise definitions of the observable uncertainties will be presented in one of the subsequent modules. Here we shall try to illustrate this principle first by discussing some measurement experiments and then present a few typical examples.

2.1 Measurement Experiments

2.1.1 Determination of Position when momentum is precisely known

Let us illustrate the uncertainty principle through a measurement experiment. Here the basic idea is to show how quantum effects intervene to prevent measurements of unlimited precision**.**Suppose we wish to determine the position of a small object with precision using a microscope. The limit to the accuracy, Δx, with which a position determination can be made is given by its resolving power

$$\Delta x = \frac{\lambda}{\sin\alpha} \tag{4.2}$$

where λ is the wavelength of the radiation that enters the objective lens and α is half the angle subtended at the position of the particle P, by the objective lens, as is shown in Fig. (4.1).

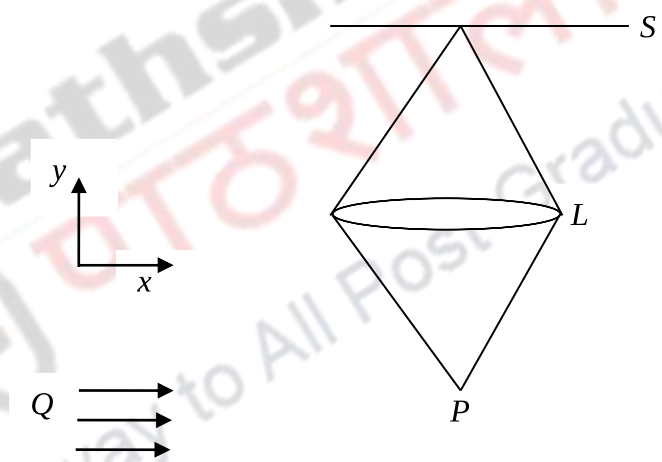

Fig. (4.1Experimental arrangement for the localization of a particle P illuminated by means of one of the scattered quanta Q. The image is focused on the screen by the lens L

Let us for simplicity focus on only one of the light quanta Q scattered by the particle P onto the screen Since the lens has a finite aperture, the direction in which the photon is scattered into lens is not precisely known. However, using the fact that the momentum p associated with wavelength λ is h / λ, the uncertainty in the x- component of the momentum, $\Delta p_x \sim \left(\frac{h}{\lambda}\right) sin\alpha$. Since in the process of scattering of the light quantum, the object P itself recoils, the x-component of the recoil momentum is also uncertain to the same extent, viz.,

$$\Delta p_x \sim \left(\frac{h}{\lambda}\right) sin\alpha = \frac{h}{\Delta x} \quad , \qquad (4.3)$$

wherein we have used Eq. (4.2). Thus, Eqn. (4.3) shows that in the very process of trying to determine position precisely, the momentum is made uncertain and the extent of this uncertainty is determined by the relation $\Delta x\, \Delta p_x \sim h$.

One way of trying to reduce the uncertainty in momentum transfer is to narrow the aperture of the lens so as to reduce the range of scattered angles. This would decrease the resolving power of the lens due to increased diffraction effects. As a result there would be a proportional decrease in the accuracy of the position measurements. Thus no matter how we perform the experiment a limit on the accuracy of the measurements corresponding to the uncertainty principle would always be introduced at some point in the experiment.

2.1.2 Momentum Determination Experiment

In the above experiment it is assumed that the momentum of the particle is precisely known before the measurement is performed. We found that the measurement not only gives an inaccurate position determination but at the same time introduces the uncertainty into the momentum.

Let us now consider a different experiment in which the position is accurately known initially and the momentum is measured. Consider the method of measuring momentum by determining the velocity by means of Doppler shift of the light radiated by the particle. (We know the velocity of a star, for example, is generally obtained by measuring the red shift.) Here the particle is, in fact, an atom when it is in an excited state. Suppose the frequency of the photon radiated by the atom, when it is at rest, is ν . Because of the Doppler effect, motion of the atom towards the observer with speed v means that the observed frequency is

$$\nu' \cong \nu\left(1+\frac{v}{c}\right) \qquad or \qquad \frac{v}{c}=\frac{\nu'-\nu}{\nu} \qquad (4.4)$$

The uncertainty in the velocity depends on the accuracy with which we measure ν′ and it is known that the uncertainty in ν′ is

$$\Delta\nu' \sim 1/\tau, \qquad (4.5)$$

where τ is the measure of uncertainty in time or the instant at which the photon is emitted; at this instant the momentum of the atom decreases by hν/c and its velocity decreases by hν/(mc). This makes the position of the atom uncertain by the amount

$$\Delta x \approx \frac{h\nu'\tau}{mc} \qquad (4.6)$$

Thus the position uncertainty arises due to the finiteness of τ.; it is because of the fact that τ is finite that we do not know when the velocity changed and therefore where the atom at later times is.

The uncertainty in momentum is obtained by the Eqs.(4.4) and (4.5):

$$\Delta p_x = m\Delta v \cong \frac{mc\Delta v}{v} \approx \frac{mc}{v\tau} \qquad (4.7)$$

Since we here consider a non-relativistic case, v/c<<1 and therefore, in this approximation, Eq.(4.4) gives $v' \cong v$. Thus, combining Eqs. (4.4) and (4.7) leads to the uncertainty $\Delta x\, \Delta p_x \approx \hbar$.

3.A Few Examples Illustrating Uncertainty Principle

Let us consider a few examples to study the implications of the uncertainty principle:

Example 1

Let us apply the first of the inequalities given above to study its implication in the case of hydrogen atom. Suppose the spread Δx in the position of the electron in hydrogen atom is of the order of its radius. Now experimentally, the size of hydrogen atom is determined to be of the order of 1 $A°$. T hus, uncertainty in the position of the electron must be of the order , $\Delta x \sim 10^{-10} m$. Using the first inequality in (4.1), would then imply the uncertainty in momentum $\Delta p \geq 10^{10}\hbar$. Since the mass of the electron is approximately $10^{-30}\, kg$, the uncertainty in the value of velocity of the electron $\geq 10^6\, m/s$, which turns out to be much higher than the known value. It is thus evident that the electron in hydrogen atom can not be described even approximately in classical terms and it makes no sense to talk of a well defined trajectory.

Example 2

In a Stern-Gerlach experiment, a beam of silver atoms emerges from an oven which contain silver vapour at a temperature 1200 °K. The beam is collimated by passing through a small aperture. Let us see how far the spot on the screen can be localized by narrowing the aperture in accordance with the uncertainty principle.

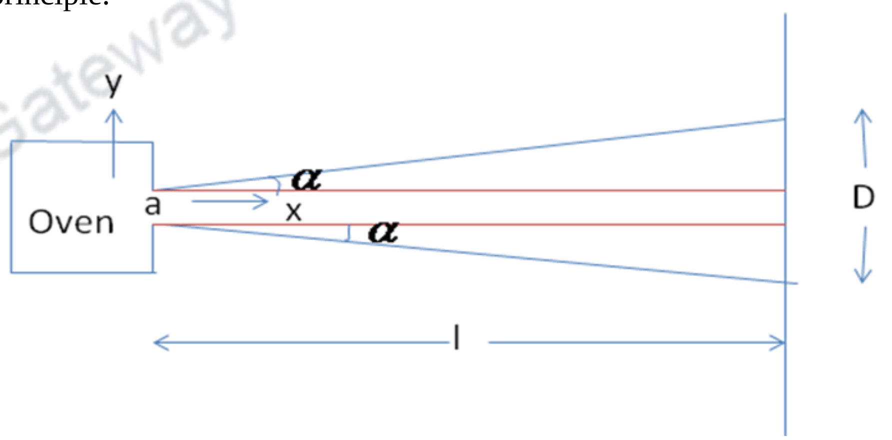

Fig. **Arrangement of Stern-Gerlach Experiment**

Suppose the atoms have a component of momentum p_x along the beam. Then by equipartition of energy,

$$\frac{p_x^2}{2m} = \frac{1}{2}kT \quad \Rightarrow \quad p_x = \sqrt{mkT} \tag{4.8}$$

The y-coordinate of an atom at the moment when it passes through the aperture is determined to a precision Δy=a, where a is the diameter of the aperture. There is therefore, an uncertainty Δp_y in the corresponding component p_y of the momentum such that $\Delta p_y \approx h/a$. This leads to a spread of the beam with angle α, where

$$\tan\alpha = \frac{\Delta p_y/2}{p_x} = \frac{h}{2a\sqrt{mkT}} \tag{4.9}$$

Now, the spot of diameter D on the screen, which is at a distance l from the oven, is given by

$$D = a + 2l\tan\alpha = a + \frac{lh}{a\sqrt{mkT}} \tag{4.10}$$

The minimum value of D of the spot is obtained by differentiating this expression with respect to a, which gives

$$a^2 = \frac{lh}{2\sqrt{mkT}} \tag{4.11}$$

Substituting this value of a, we get

$$D_{\min} = \frac{3}{\sqrt{2}} \frac{\sqrt{lh}}{[mkT]^{1/4}} \tag{4.12}$$

Since the right hand side of this expression has all fixed constants, it is clear one can not minimize the spot on the screen beyond a certain value.

Example 3:

It is believed that the origin of fundamental forces of Nature taking place between different constituents is due to the exchanged quanta. One can make use of the uncertainty principle, $\Delta E\,\Delta t \approx h$, to estimate the rest mass of the exchanged quanta for the given range of interaction. The range of the interaction is related with the energy E of the field quantum exchanged. The exchanged quantum, being virtual, exists only for a short time t $\leq$ h/E. Assuming that the field quanta travel with velocity of light c, the range

$$r_0 = ct \approx \frac{hc}{E}. \tag{4.13}$$

If the field quantum has rest mass m, it must have at least , energy, E=mc^2. Using this value of E in the above relation gives

$$r_0 \approx \frac{h}{mc} \qquad or \qquad m \approx \frac{h}{r_0 c} \tag{4.14}$$

This relation clearly shows that if the range of interaction is infinite, the mass of the exchanged quantum is zero as is known in the case of **electromagnetic or gravitational forces**. For a finite range, as in the case of nuclear force, if we take the value of $r_0 \cong 1.4\, fm,$ the mass of the exchanged quantum , using the above relation, gives the value to be about 140 MeV/c^2. This

particle is identified as pion, originally proposed by Yukawa as a mediator responsible for nuclear interaction occurring between nucleons.

Example 4:

Consider, in a Frank-Hertz experiment, a mono-energetic beam of electrons incident on hydrogen atoms. As a result these atoms are raised to their first excited state by colliding with a beam of electrons. Suppose mean life of the excited hydrogen atoms is τ. This means that these atoms on an average take a time τ to radiate their energy and return to the ground state. The excited state has therefore a life time with a precision Δt$\approx \tau$ and therefore its energy in accordance with the uncertainty principle will have an uncertainty ΔE$\geq \frac{\hbar}{\tau}$. Due to short life time, the energy fluctuations in the energy of the excited atoms are thus appreciable. In fact it is the energy fluctuations ΔE due to the short lives of excited states which give rise to the **natural widths of spectral lines.**

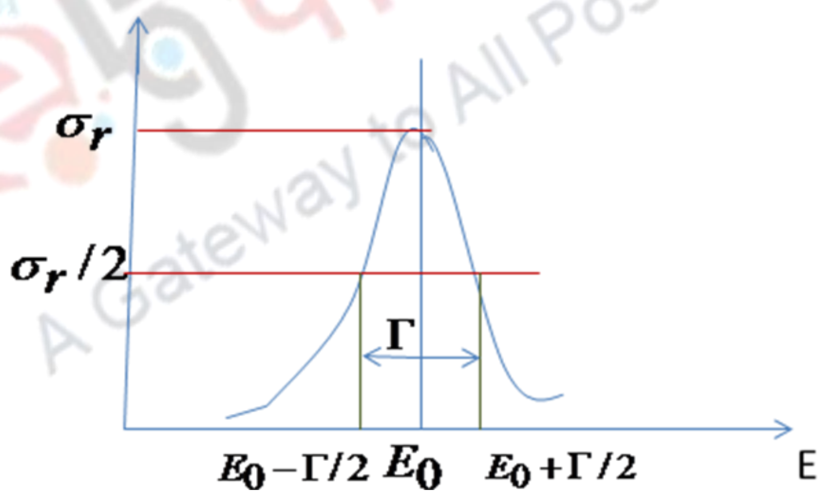

Fig. Spectral ine of an excited state having a width due to energy fluctuation

4.Complementarity Principle :

Let us revisit Young's double slit experiment showing the interference of light (or , equivalently of a beam of electrons instead of light) in the context of uncertainty principle, and probe the question: can we say that the photon (or electron) which reaches some point of the interference pattern has followed a definite trajectory passing through a particular one of the two slits? The probability interpretation, which was earlier introduced, makes it abundantly clear that we cannot. This is because for interference to take place, a particle has a definite probability where the wave function is non-vanishing and the wave function has to have non-vanishing values in a region which includes location of both the slits.

We can view it in a slightly different way assuming that the particle has a trajectory. Because of the wave nature of the particle, a sharp trajectory is not possible. Let us suppose that the trajectory is a moving wave packet with lateral extension less than half the separation 'a' between the two slits. This assumption is made to ensure that it can go through only one of the two slits. Can there be an interference pattern then? Let us try to answer this question from the point of view of uncertainty principle. As we have assumed that the extension of the wave packet is less than a/2, the uncertainty in a component of the position vector perpendicular to the direction(assumed to be along x-axis) of motion of the particle is $\Delta y < a/2$. Therefore, according to the uncertainty principle, the corresponding transverse component of momentum has uncertainty, Δp_y , i.e., at least (h / Δy) = 2h / a. Therefore the angle, θ, giving the direction of motion of the particle is uncertain by an amount

$$\theta \approx \tan(\theta) = \left(\frac{\Delta p_y}{p}\right) = \frac{2h}{pa} = \frac{2\lambda}{a} , \tag{4.15}$$

where λ is the de- Broglie wave length of the particle. But we know the angular separation between successive maxima of the interference pattern in Young's double slit is λ/a. Clearly, as the above uncertainty exceeds this value, the interference pattern will be washed out. Note that the above analysis holds for a material particle as well as for a photon.

The situation analyzed above clearly shows that when the wave packet is made compact enough to make it behave like a particle, it then loses the ability to display wave properties. In other words, **the particle and the wave aspect of a physical entity are complementary and cannot be demonstrated at the same time. This is the complementary principle of Bohr.**

Bohr, thus, introduced this principle to understand in more physical terms the implications of Heisenberg's uncertainty principle. According to this, atomic phenomena can not be described with the precision demanded by classical dynamics: the complementary elements which are necessary for a complete description are mutually exclusive. Looking from an experimentalist's point of view, this uncertainty or inaccuracy in measurement is not because of any deficiency in his skill or because of the limitation in the apparatus available to him but rather my be regarded as a law of nature that measurements more precise than those demanded by the uncertainty principle can not be made.

5. SUMMARY

After studying this module, you should be able to

- Know the statement of Heisenberg's Uncertainty Principle
- Learn through a measurement experiment how the process of trying to determine position precisely using a precise value of momentum, not only gives an inaccurate position determination but at the same time introduces the uncertainty into the momentum.
- Learn through another measurement experiment that even if the position is known precisely the process of measuring momentum not only gives uncertainty in the momentum measurement but at the same time introduces uncertainty in its position
- Know through a few typical examples the implications /consequences of using the uncertainty principle
- Learn the complementary principle given by Bohr

Similarly, superposition of a number of harmonic waves of varying frequencies or angular frequencies can produce a localized wave packet in time. We shall also analyze the evolution of a wave packet in time and show that the spread of the wave packet increases with time.