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MHRD Govt. of India

An MHRD Project under its National Mission on Education throught ICT (NME-ICT)

Subject: Physics

Production of Courseware

e-Content for Post Graduate Courses

Paper No. : Quantum Mechanics-I

Module : Angular Momentum

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Description of Module	
Subject Name	Physics
Paper Name	Quantum Mechanics-I
Module Name/Title	Angular Momentum
Module Id	23

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1. Learning Outcomes

After studying this module, you shall be able to

- Define orbital angular momentum operator and learn the commutation relations between its components
- Identify angular momentum as a generator of infinitesimal rotations in three-dimensional space
- Learn that the origin of commutation relations lies in the geometric properties of rotations and know the need to generalize the definition of angular momentum
- show that the commutation relations determine the quantal properties of the angular momentum, i.e., the eigenvalues and eigenvectors of the angular momentum operator
- learn the significance of ladder operators or what are called raising and lowering operators

2. Introduction

The study of angular momentum assumes greater importance in quantum mechanics than even in classical mechanics. There could be many reasons for this. In the first place, you know from the study of Stern-Gerlach experiment that angular momentum is quantized; second is the observation for the existence of intrinsic spin (angular momentum) of elementary particles like electrons, protons etc., third is the importance of periodic motions, envisaged as motion in closed orbits involving angular momentum. A thorough understanding of angular momentum is, therefore, essential in atomic, molecular and nuclear spectroscopy. Even in scattering and collision problems, considerations of angular momentum play important role. Angular momentum concepts have led to natural generalizations, as for instance, isospin in nuclear physics and SU(3) symmetries in particle physics.

3. Angular Momentum

3.1 Definition of Angular Momentum

As studied in classical mechanics, the angular momentum of a particle about a point O, shown in Fig. (23.1), is defined as

$$\vec{L} = \vec{r} \times \vec{p}\,, \tag{23.1}$$

where $\vec{r}$ is the position vector and $\vec{p}$ is the linear momentum of the particle.

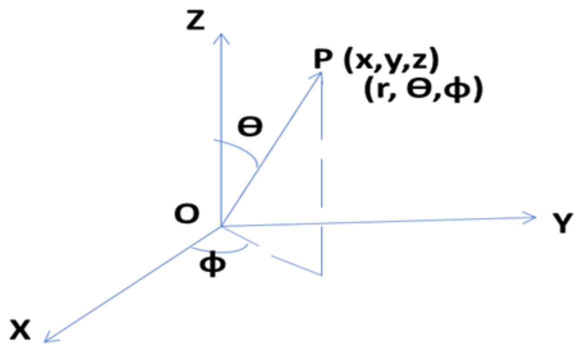

Fig. 23.1

The quantum mechanical operator for the angular momentum is obtained by replacing the dynamical variables $\vec{r}$ and $\vec{p}$ by the corresponding operators. Thus

$$\hat{L}_x = (\hat{y}\,\hat{p}_z - \hat{z}\,\hat{p}_y),$$
$$\hat{L}_y = (\hat{z}\,\hat{p}_x - \hat{x}\,\hat{p}_z), \qquad (23.2)$$
$$\hat{L}_z = (\hat{x}\,\hat{p}_y - \hat{y}\,\hat{p}_x)$$

Alternatively, we can also write, using the notation, x≡ x_1 , y≡ x_2 , etc,

$$\hat{L}_i = \epsilon_{ijk}\,\hat{x}_j\,\hat{p}_k\,;$$

where a repeated index is to be summed. Here ,

$$\epsilon_{ijk} = +1, \quad \text{if i, j, k is even permutation } of\ 1,2,3$$

and

$$\epsilon_{ijk} = -1, \quad \text{if i, j, k is an odd permutation } of\ 1,2,3 \qquad (23.3)$$
$$\epsilon_{ijk} = 0, \quad \text{if any two indices are equal.}$$

By using the commutation relations:

$$[\hat{x}_i\,,\hat{p}_j] = i\hbar\delta_{ij}, \quad [\hat{x}_i\,,\hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \ , \qquad (23.4)$$

It is a simple exercise to show that

$$[\hat{L}_i\,,\hat{L}_j] = i\hbar\,\epsilon_{ijk}\,\hat{L}_k \qquad (23.5)$$

Also,

$$[\vec{\hat{L}}^2, \hat{L}_k] = 0, \qquad (k = 1,2,3), \qquad (23.6)$$

where

$$\vec{\hat{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \qquad (23.7)$$

This shows that while the components of angular momentum operators do not commute with each other, the square of the total angular momentum operator, $\hat{L}^2$, on the other hand, commutes with each component.

3.1.1 Angular Momentum as the Generator of Infinitesimal Rotations

To get physical insight into the nature of the orbital angular momentum, let us analyze its connection with rotations in space.

Consider the case of a change of representation induced by a rotation of coordinate system, from S (OXYZ) to S' (O$X'Y'Z'$) obtained by rotating the axes through an angle θ about OZ. As a result, a point whose coordinates in S are $\vec{r}\,(x,y,z)$ will have coordinates $\vec{r}'(x',y',z')$ in S', given by

$$x' = x cos\theta + y sin\theta$$
$$y' = -x sin\theta + y cos\theta \qquad (23.8)$$
$$z' = z$$

The wave function ψ' describing the rotated state at $\vec{r}'$ must be equal to the original wave function ψ at the point $\vec{r}$, i.e.,

$$\psi'(\vec{r}') = \psi(\vec{r}) \qquad (23.9)$$

On substituting for $\vec{r}'$ from Eq.(23.8) and assuming θ to be infinitesimally small, we write $\psi'(x + y\theta, y - x\theta, z) = \psi(x,y,z)$. It is, however, more convenient if we replace x by x-$y\theta$ and y by y+xθ on both sides of the equation. We thus have

$$\psi'(x,y,z) = \psi(x - y\,\theta, y + x\,\theta, z) \qquad (23.10)$$

Using the Taylor's series expansion for ψ on the left hand side and retaining only the first term in the expansion, we get

$$\psi'(x,y,z) = \psi(x,y,z) + \theta\left\{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right\}\psi(x,y,z) \qquad (23.11)$$

We note that the differential operator appearing in Eq.(23.11) can just be identified as the z-component of the angular momentum, $\hat{L}_z$, i.e.,

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \qquad (23.12)$$

It thus enables us to write Eq.(23.11) as

$$\psi'(x,y,z) = \left[1 + i\theta\ {}^{\hat{L}_z}/_{\hbar}\right]\psi(x,y,z) \qquad (23.13)$$

It is, therefore, said that ${}^{\hat{L}_z}/_{\hbar}$ plays the role of the generator of infinitesimal rotations. In general, if the rotation is about an arbitrary axis along the unit vector $\hat{n}$, Eq.(23.13) is then written as

$$\psi'(\vec{r}) = \left[1 + \frac{i\theta\hat{n}.\vec{L}}{\hbar}\right]\psi(\vec{r}) \qquad (23.14)$$

For a rotation through a finite angle θ, we sum up the series, instead of retaining only the first term , and get

$$\psi'(\vec{r}) = \exp[i\theta\frac{\hat{n}.\vec{L}}{\hbar}]\,\psi(\vec{r}) \qquad (23.15)$$

The important point to note from this study is that the origin of the commutation relations, Eqs.(23.5) and (23.6), lies in the geometric properties of rotations in three dimensional space and that these relations are very general, including those of $\hat{L}_x, \hat{L}_y$ and $\hat{L}_z$ as a special case. We therefore adopt a more general point of view and define an angular momentum $\vec{\hat{J}}$, with three components, $\hat{J}_x, \hat{J}_y$ and $\hat{J}_z$, satisfying the commutation relations:

$$[\hat{J}_x, \hat{J}_y] = i\hbar\ \hat{J}_z\ , \qquad [\hat{J}_y, \hat{J}_z] = i\hbar\ \hat{J}_x\ , \qquad [\hat{J}_z, \hat{J}_x] = i\hbar\ \hat{J}_y \qquad (23.16)$$

We also introduce the operator

$$\vec{\hat{J}}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \qquad (23.17)$$

It is a simple exercise to show that $\hat{J}^2$ commutes with each of the components, $\hat{J}_x, \hat{J}_y$ and $\hat{J}_z$, i.e.,

$$[\hat{J}^2, \hat{J}_x] = [\hat{J}^2, \hat{J}_y] = [\hat{J}^2, \hat{J}_z] = 0 \qquad (23.18)$$

3.2 Eigen Values and Eigen Vectors

We shall now show that the commutation relations, Eq.(23.16), determine the quantal properties of the angular momentum, i.e., the eigenvalues and eigenvectors of the angular momentum operator are determined completely by Eq.(23.16) and the general properties of the Hilbert space. Since the components of the angular momentum do not commute among themselves, one can not find a common basis for all the three components. However, because $\hat{J}^2$ commutes with $\vec{\hat{J}}$, we can have a common basis for $\hat{J}^2$ and any one component, say $\hat{J}_z$, of $\vec{\hat{J}}$. Also note that the eigen values of $\hat{J}^2$ and

$\hat{J}_z$, are real, since they are Hermitian operators. It is, therefore possible to find a complete set of simultaneous eigen states of $\hat{J}^2$ and any one

component of $\vec{\hat{J}}$, which we choose to be $\hat{J}_z$. Let us represent such a common basis by the state $|jm\rangle$, where j labels the eigen values of $\hat{J}^2$ and m those of $\hat{J}_z$. The state vectors $|jm\rangle$ are orthonormal, i.e.,

$$\langle j'm'|jm\rangle = \delta_{jj'}\delta_{mm'} \qquad (23.19)$$

By definition,

$$\hat{J}^2|jm\rangle = \hbar^2\lambda_j|jm\rangle \qquad (23.20a)$$

$$\hat{J}_z|jm\rangle = \hbar m|jm\rangle \qquad (23.20b)$$

Note that $\hat{J}^2$, being the sum of the square of Hermitian operators, is positive definite. Therefore,

$$\lambda_j \equiv \frac{\langle jm|\hat{J}^2|jm\rangle}{\hbar^2} \geq 0\,. \qquad (23.21)$$

It is also clear that

$$\langle jm|\hat{J}^2|jm\rangle \equiv \left\langle \hat{J}^2 \right\rangle = \left\langle \hat{J}_x^2 \right\rangle + \left\langle \hat{J}_y^2 \right\rangle + \left\langle \hat{J}_z^2 \right\rangle \geq \left\langle \hat{J}_z^2 \right\rangle, \qquad (23.22)$$

i.e., $$\lambda_j\hbar^2 = \left\langle \hat{J}_x^2 \right\rangle + \left\langle \hat{J}_y^2 \right\rangle + m^2\hbar^2$$

Since $\hat{J}_x$ *and* $\hat{J}_y$ are Hermitian, $\hat{J}_x^2$ *and* $\hat{J}_y^2$ are positive operators and therefore their expectation values are necessarily positive. It follows that

$$\lambda_j \geq m^2 \geq 0 \qquad (23.23)$$

3.2.1 Ladder Operators:

At this stage it is convenient to introduce the operators

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \qquad \textbf{and} \qquad \hat{J}_- = \hat{J}_x - i\hat{J}_y \qquad (23.24)$$

The commutation relations, (23.16), can readily be expressed in terms of $\hat{J}_+$, $\hat{J}_-$ **and** $\hat{J}_z$ as

$$[\hat{J}_z, \hat{J}_+] = \hbar\hat{J}_+ \qquad (23.25a)$$

$$[\hat{J}_z, \hat{J}_-] = -\hbar\hat{J}_- \qquad (23.25b)$$

$$[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_z \qquad (23.25\ c)$$

Note that $\hat{J}_+$ **and** $\hat{J}_-$ are not Hermitian operators; instead

$$\hat{J}_+^\dagger = \hat{J}_- \qquad \textbf{and} \qquad \hat{J}_-^\dagger = \hat{J}_+ \qquad (23.26)$$

The square of the angular momentum operator can now be expressed in the following forms:

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 \qquad (23.27a)$$

$$= \hat{J}_-\hat{J}_+ + \hbar\hat{J}_z + \hat{J}_z^2 \qquad (23.27b)$$

$$= \hat{J}_+\hat{J}_- - \hbar\hat{J}_z + \hat{J}_z^2 \qquad (23.27c)$$

This can be easily seen if we remember that

$$\hat{J}_-\hat{J}_+ = (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y) = \hat{J}_x^2 + \hat{J}_y^2 + i\hat{J}_x\hat{J}_y - i\hat{J}_y\hat{J}_y$$

$$= \hat{J}_x^2 + \hat{J}_y^2 - \hbar \hat{J}_z \quad (23.28)$$

Similarly, one can find

$$\hat{J}_+ \hat{J}_- = \hat{J}_x^2 + \hat{J}_y^2 + \hbar \hat{J}_z \quad (23.29)$$

Using these relations, it is simple to express Eq.(23.27a) in the forms given in Eqs.(23.27b) and (23.27c).

Since $\hat{J}_+$ **and** $\hat{J}_-$ commute with $\hat{J}^2$, $\hat{J}_\pm |jm\rangle$ are eigen vectors of $\hat{J}^2$having the same eigen value as that of $|jm\rangle$, i.e.,

$$\hat{J}^2 (\hat{J}_\pm |jm\rangle) = \hbar^2 \lambda_j (\hat{J}_\pm |jm\rangle) \quad (23.30)$$

But $\hat{J}_z$ operating on $\hat{J}_\pm |jm\rangle$ can be worked out , using the commutation relations,Eqs.(23.25a) and (23.25b) to give

$$\hat{J}_z (\hat{J}_\pm |jm\rangle) = (\hat{J}_\pm \hat{J}_z \pm \hbar \hat{J}_\pm)|jm\rangle = (m \pm 1)\hbar(\hat{J}_\pm |jm\rangle) \quad (23.31)$$

This shows $\hat{J}_+ |jm\rangle$ is an eigen vector of $\hat{J}_z$ belonging to the eigen value (m+1) $\hbar$, whereas $\hat{J}_- |jm\rangle$ is an eigen vector of $\hat{J}_z$ with eigen value (m-1) $\hbar$. On comparing with Eqs.(23.20a) and (23.20b), we conclude that $\hat{J}_+ |jm\rangle$ gives a state $|j, m+1\rangle$ apart from a possible normalization constant, while $\hat{J}_- |jm\rangle$ is proportional to $|j, m-1\rangle$. We thus write

$$\hat{J}_+ |jm\rangle = c_{jm}^+ \hbar |jm+1\rangle \quad (23.32)$$

$$\hat{J}_- |jm\rangle = c_{jm}^- \hbar |jm-1\rangle \ , \quad (23.33)$$

where c_{jm}^+ *and* c_{jm}^- are the normalization constants to be determined.

From Eqs.(23.32) and (23.33), it is important to notice that while the effect of the operator $\hat{J}_+$ on the state $|j, m\rangle$ is to increase the eigen value of $\hat{J}_z$ by one unit, i.e., it raises the state to $|j, m+1\rangle$, the operator $\hat{J}_-$ acting on the state $|j, m\rangle$ lowers the state to $|j, m-1\rangle$, resulting in reducing the eigen value of $\hat{J}_z$ by one unit. It is for this reason that the operators $\hat{J}_+$ ***and*** $\hat{J}_-$ are respectively known as raising and lowering operators or termed as ladder operators.

The repeated operation of the operator $\hat{J}_+$ on Eq.(23.32) shows that the given state $|j, m\rangle$ would go on increasing the index m to m+1, m+2, m+3 ….., similarly the operation $\hat{J}_-$ would be lowering it to m-1, m-2, m-3,…… But clearly, the series must terminate, otherwise we would have vectors like $|j, m'\rangle$ which violate the inequality, Eq.(23.23), since λ_j is not changed by the application of $\hat{J}_\pm$ on $|j, m\rangle$. This means in order to terminate the series, there must exist a maximum value, say m_2 for which $\hat{J}_+ |jm_2\rangle = 0$. Similarly, there must be a minimum value, say m_1 , for which $\hat{J}_- |jm_1\rangle = 0$.

Since the states $|jm_2\rangle$ *and* $|jm_1\rangle$ are obtained from $|jm\rangle$ by repeated application of $\hat{J}_+$ *and* $\hat{J}_-$ respectively, we must have a condition

$$m_2 - m_1 = \text{a positive integer or zero} \qquad (23.34)$$

Now the state vector $\hat{J}_+|jm_2\rangle = 0$ implies that its norm is zero, that is

$$(\hat{J}_+|jm_2\rangle)^*(\hat{J}_+|jm_2\rangle) = 0$$
$$or \quad \langle jm_2|\hat{J}_-\hat{J}_+|jm_2\rangle = 0 \qquad (23.35)$$

Using the Eq.(23.28), i.e., $\hat{\mathbf{J}}_-\hat{\mathbf{J}}_+ = \hat{\mathbf{J}}_x^2 + \hat{\mathbf{J}}_y^2 - \hbar\,\hat{\mathbf{J}}_z = \hat{\boldsymbol{J}}^2 - \hat{\boldsymbol{J}}_z(\hat{\boldsymbol{J}}_z + \hbar)$ in Eq.(23.35) , and using Eqs.(23.20a) and (23.20b), we get

$$\lambda_j = m_2(m_2+1) \qquad (23.36)$$

Similarly, the equation $\hat{J}_-|jm_1\rangle = 0$ gives

$$\lambda_j = m_1(m_1-1) \qquad (23.37)$$

Equating Eqs.(23.36) and (23.37), we have

$$m_2(m_2+1) = m_1(m_1-1)$$
$$or \quad (m_2+m_1)(m_2-m_1+1) = 0 \qquad (23.38)$$

This shows that either $m_2 - m_1 = -1$ *or* $m_2 = -m_1$. Out of the two, first one is ruled out because of the condition (23.34) and therefore only the second, *according to which the minimum value is the negative of the maximum,* is an acceptable solution.

Now, λ_j, being the eigen value of $\hat{\boldsymbol{J}}^2$, depends only on j (cf., Eq.(23.20a)) so that from Eqs.(23.36) or (23.37), one concludes that m_2 *and* m_1 should be functions only of j. The choice $m_2 = j$ *and* $m_1 = -m_2 = -j$ *leading to* the requirement

$$m_2 - m_1 = 2j = \text{ a positive integer or zero}$$

meets these conditions. As $m_2 - m_1$ can be zero or a positive integer, j can have the values

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, \ldots\ldots\ldots\ldots\ldots + \infty. \qquad (23.39a)$$

According to Eq.(23.36), the eigen value λ_j is given by

$$\lambda_j = j(j+1) \qquad (23.40)$$

and m can take the values : m= - j to + j , differing by integer, since m can change by integer only, i.e

$$m = -j, -j+1, -j+2, \ldots\ldots \quad, +j-2, +j-1, +j \qquad (23.39b)$$

We have thus obtained the eigen value spectrum of angular momentum operator $\vec{\boldsymbol{J}}$, just starting from the commutation relations between the components of angular momentum and the basic postulates of quantum mechanics,. The remarkable point to note is that half integral values of j have emerged from this general treatment in a natural way. As we know, half integral values of j are possible only when spin is involved. For a given value of j, there are (2j+1) linearly independent state vectors corresponding to the (2j+1) different values of m given by (23.39b). They are the common eigen vectors of $\hat{J}^2$ and $\hat{J}_z$.

4. Summary

In this module, you have learnt to

- Define orbital angular momentum operator and write the commutation relations between its components
- Identify angular momentum as a generator of infinitesimal rotations in three-dimensional space
- Know that the origin of commutation relations lies in the geometric properties of rotations and the need to generalize the definition of angular momentum
- show that the commutation relations determine the quantal properties of the angular momentum, i.e., the eigenvalues and eigenvectors of the angular momentum operator
- know the significance of ladder operators or what are also called raising and lowering operators.